DIFFRACTION OF AN ELASTIC PLANE WAVE AT A MASSIVE STRIP SOLDERED INTO INFINITE ELASTIC MEDIUM

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The motion of an obstacle having the form of an infinite strip and acted upon by a plane elastic wave was considered in [1]. This paper deals with disturbances of a field of incident wave caused by an obstacle (*).

1. We will adopt a system of units of measurement in which the strip width, the density of the medium and the velocity of the transverse waves equal unity, so that the medium is characterized by a single parameter γ which is the ratio of the transverse and longitudinal wave velocities.

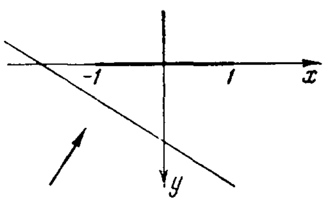

Fig. 1

The system of Cartesian coordinates is chosen so that the strip (Fig.1) occupies a part of the plane $y = 0$ for $|x| \leqslant 1, -\infty < z < \infty$ and all the quantities are independent of the z-coordinate.

We will decompose the displacement field of the medium into the undisturbed field of the incident wave with displacement components $u_i(t - \vartheta(x+1) + \delta y)$ and $v_i(t - \vartheta(x+1) + \delta y)$ along the x and y axes, respectively, and the disturbance with displacement components $u(x, y, t)$, $v(x, y, t)$, which contains waves diffracted at the edges of the strip,

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*) It is the author's fault that Formula (4.11) on page 107 in [1] contains errors. The correct expression should read

$$M_j(t) = \vartheta^{-2}\nu_j\{M_{(2)}(\lambda)\,[\lambda k_{(2)1} - (1 + \lambda\nu_j)\,k_{(2)0}]\,\delta(t) +$$

$$+ {}^1\!/_2\,\lambda^2\nu_j k_{(2)0}^2 S_{(2)12}(t - 2\gamma, \lambda)\} * v_i(t - \vartheta - \lambda)\,\Big|_{\lambda=-\vartheta}^{\lambda=\vartheta} -$$

$$- \nu_j k_{(2)0}\,[(k_{(2)0}\nu_j - k_{(2)1})\,S_{(2)12}(t, 0) + k_{(2)0}S_{(2)22}(t, 0)] * \alpha(t - 2\gamma)$$

Formula (5.6) will change correspondingly

$$M_j(t) = \vartheta^{-2}\,[\vartheta k_{(2)1} + (1 - \vartheta\nu_j)\,k_{(2)0}]\,\nu_j K_{(2)}(-\vartheta)\,v_i(t) +$$

$$+ \vartheta^{-2}\,[\vartheta k_{(2)1} - (1 + \vartheta\nu_j)\,k_{(2)0}]\,\nu_j M_{(2)}(\vartheta)\,v_i(t - 2\vartheta)$$

as well as those generated by motion of the strip. Rather than to describe the field of disturbance under consideration by the components of the displacement vecor, it is more convenient to present it in terms of corresponding potentials, transverse φ and longitudinal ψ, bearing in mind that

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{1.1}$$

The boundary conditions [1] have the form

$$\begin{gathered} u = -u_i(t-(x+1)\vartheta) + u_0(t) \\ v = -v_i(t-(x+1)\vartheta) + v_0(t) + (x-x_0)\,\alpha(t) \\ \text{for } |x| \leqslant 1,\ y = 0 \end{gathered} \tag{1.2}$$

where (u_0, v_0) is the displacement vector of the strip's centroid, x_0 is its coordinate in the equilibrium position, α is the angle of rotation of the strip. Hence, we easily conclude, that the unknown disturbance is made up of two parts, one describing the diffraction of the incident wave at the edges of the strip which may be regarded as fixed for the purpose of computing this portion of the disturbance ($u_0 = v_0 = \alpha = 0$), and the other part describing waves generated by motion of the strip, in which case we may set $u_i = v_i = 0$ (after the elements of motion of the strip are computed in [1], we can consider them to be given independently of the incident wave; for the same reason the mass of the strip and its moment of inertia are not mentioned in this paper).

2. From formulas found in [1] one can derive the following expressions relating the double Laplace transforms of the longitudinal and transverse potentials to the double Laplace transforms of the jumps in stress components on the strip

$$\begin{gathered} \varphi(q,y,p) = \left[-\frac{q\tau_{(1)}^{\circ}(q,p)}{p^2\sqrt{\gamma^2p^2-q^2}} + \operatorname{sgn} y\,\frac{\sigma_{(2)}^{\circ}(q,p)}{p^2}\right]\exp\left(-|y|\sqrt{\gamma^2p^2-q^2}\right) \\ \psi(q,y,p) = \left[\operatorname{sgn} y\,\frac{\tau_{(1)}^{\circ}(q,p)}{p^2} + \frac{q\sigma_{(2)}^{\circ}(q,p)}{p^2\sqrt{p^2-q^2}}\right]\exp\left(-|y|\sqrt{p^2-q^2}\right) \end{gathered} \tag{2.1}$$

where $\sigma_{(2)}^{\circ}(q,p)$ and $\tau_{(1)}^{\circ}(q,p)$ are the Laplace transforms of the jumps of the normal and shear stresses respectively. The potentials are expressed in terms of their transforms by the inversion formula

$$f(x,y,t) = -\frac{1}{4\pi^2}\int_{-i\infty+c}^{i\infty+c} e^{pt}\int_{-i\infty+c'}^{i\infty+c'} e^{qx} f(q,y,p)\,dq\,dp \tag{2.2}$$

where $c > 0$ and c' should be chosen so that the path of integration with respect to q lies entirely in the domain of regularity of $f(q,y,p)$. Here the transforms are designated by the same letters as the original functions, the difference shown only in the arguments. The quantities $\tau_{(1)}^{\circ}(q,p)$ and $\sigma_{(2)}^{\circ}(q,p)$ have been computed in [1] and are given by the formulas (here we write down only the terms describing diffraction)

$$\begin{gathered} K_{(l)}(-\vartheta)\left\{e^{q}K_{(l)}\left(-\frac{q}{p}\right)\left(\vartheta+\frac{q}{p}\right)^{-1} - \right. \\ -\sum_{k=1}^{\infty} K_{(l)}\left((-1)^{k+1}\frac{q}{p}\right)\exp\left[(-1)^k q\right]\ \text{v. p.}\int_{\gamma}^{\infty}\cdots\int_{\gamma}^{\infty}\Pi_{(l)k}(\zeta)\,\times \\ \left.\times\frac{d\Omega_\zeta}{(\zeta_1-\vartheta)\left[\zeta_k-(-1)^k q/p\right]}\right\} - M_{(l)}(\vartheta)\,e^{-2p\vartheta}\left\{e^{-q}K_{(l)}\left(\frac{q}{p}\right)\left(\vartheta+\frac{q}{p}\right)^{-1} + \right. \\ +\sum_{k=1}^{\infty} K_{(l)}\left((-1)^k\frac{q}{p}\right)\exp\left[(-1)^{k+1}q\right]\int_{\gamma}^{\infty}\cdots\int_{\gamma}^{\infty}\Pi_{(l)k}(\zeta)\,\times \\ \left.\times\frac{d\Omega_\zeta}{(\zeta_1+\vartheta)\left[\zeta_k+(-1)^k q/p\right]}\right\} = \begin{cases}\tau_{(1)}^{\circ}(q,p)/u_i(p) & \text{for } l=1\\ \sigma_{(2)}^{\circ}(q,p)/v_i(p) & \text{for } l=2\end{cases} \end{gathered} \tag{2.3}$$

Here

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k), \qquad d\Omega_\zeta = d\zeta_1 \ldots d\zeta_k$$

$$\Pi_{(l)k}(\zeta) = \frac{1}{\pi^k} L_{(l)}(\zeta_1) \left[\prod_{j=2}^{k} \frac{L_{(l)}(\zeta_j)}{\zeta_j + \zeta_{j-1}} e^{-2p\zeta_j} \right] e^{-2p\zeta_1} \tag{2.4}$$

$$K_{(1)}(\sigma) = \frac{\sqrt{2(\gamma - \sigma)}}{\sqrt{1+\gamma^2}} e^{g(\sigma)}, \qquad K_{(2)}(\sigma) = \frac{\sqrt{2(1-\sigma)}}{\sqrt{1+\gamma^2}} e^{g(\sigma)} \tag{2.5}$$

$$g(\sigma) = \frac{1}{\pi} \int_\gamma^1 \tan^{-1} \left[\left(1 - \frac{\gamma^2}{\zeta^2}\right)\left(\frac{1}{\zeta^2} - 1\right)\right]^{1/2} \frac{d\zeta}{\zeta - \sigma} \tag{2.6}$$

The square roots in the above expressions are uniform by taking the branch cuts in the σ-plane along the segments of the real axis $[\pm\gamma, \pm\infty)$ and choosing the branch which is positive for $\sigma = 0$. The functions $M_{(l)}(\sigma)$ and $L_{(l)}(\sigma)$ are defined only for real values of σ and are:

$$M_{(l)}(\sigma) = \mathrm{Re}\, K_{(l)}(\sigma - i0), \qquad L_{(l)}(\sigma) = \mathrm{Im}\, \frac{K_{(l)}(\sigma - i0)}{K_{(l)}(-\sigma)} \tag{2.7}$$

Although Expressions (2.3) are already quite complicated, it makes sense to complicate them some more in order to ascribe a clear physical meaning to each term. With that purpose in mind let us present the functions $L_{(l)}(\sigma)$ as sums $L_{(l)p}(\sigma) + L_{(l)s}(\sigma)$, where

$$\begin{aligned}
L_{(1)p}(\sigma) &= \frac{\sigma^2 \sqrt{\sigma^2 - \gamma^2}\,(1+\gamma^2)}{2[(1+\gamma^2)\sigma^2 - \gamma^2](\gamma + \sigma)} e^{-2g(-\sigma)} && (\sigma > \gamma) \\
L_{(2)p}(\sigma) &= \frac{(\sigma - 1)\sqrt{\sigma^2 - \gamma^2}\,(1+\gamma^2)}{2[(1+\gamma^2)\sigma^2 - \gamma^2]} e^{-2g(-\sigma)} && (\sigma > \gamma) \\
L_{(1)s}(\sigma) &= \frac{(\sigma - \gamma)\sqrt{\sigma^2 - 1}\,(1+\gamma^2)}{2[(1+\gamma^2)\sigma^2 - \gamma^2]} e^{-2g(-\sigma)} && (\sigma > 1) \\
L_{(2)s}(\sigma) &= \frac{\sigma^2 \sqrt{\sigma^2 - 1}\,(1+\gamma^2)}{2[(1+\gamma^2)\sigma^2 - \gamma^2](1+\sigma)} e^{-2g(-\sigma)} && (\sigma > 1) \\
L_{(l)p}(\sigma) &= 0 \quad (\sigma < \gamma), \qquad L_{(l)s}(\sigma) = 0 && (\sigma < 1)
\end{aligned} \tag{2.8}$$

Now we can put $\Pi_{(l)k}$ into the form

$$\Pi_{(l)k} = \sum_{(r)} \Pi_{(l)(r)k} \tag{2.9}$$

where (r) is a composite index which gives the sequence of indices p and s of functions L, contained in each term. For instance, when $k = 3$ we have

$$\begin{aligned}
\Pi_{(l)3}(\zeta) &= \Pi_{(l)ppp3} + \Pi_{(l)pps3} + \Pi_{(l)psp3} + \\
&+ \Pi_{(l)spp3} + \Pi_{(l)pss3} + \Pi_{(l)sps3} + \Pi_{(l)ssp3} + \Pi_{(l)sss3} = \\
&= \pi^{-3} L_{(l)p}(\zeta_1) L_{(l)p}(\zeta_2) L_{(l)p}(\zeta_3)(\zeta_1 + \zeta_2)^{-1}(\zeta_2 + \zeta_3)^{-1} e^{-2p(\zeta_1+\zeta_2+\zeta_3)} + \\
&+ \pi^{-3} L_{(l)p}(\zeta_1) L_{(l)p}(\zeta_2) L_{(l)s}(\zeta_3)(\zeta_1 + \zeta_2)^{-1}(\zeta_2 + \zeta_3)^{-1} e^{-2p(\zeta_1+\zeta_2+\zeta_3)} + \\
&+ \ldots + \pi^{-3} L_{(l)s}(\zeta_1) L_{(l)s}(\zeta_2) L_{(l)s}(\zeta_3)(\zeta_1 + \zeta_2)^{-1}(\zeta_2 + \zeta_3)^{-1} e^{-2p(\zeta_1+\zeta_2+\zeta_3)}
\end{aligned}$$

It is easy to see that for any k the function $\Pi_{(l)k}$ is made up of 2^k terms. After the substitution of (2.9) into (2.3) and then into (2.1), the potentials are expressed in the form of double sums, in which a definite physical meaning can be ascribed to each term. Namely, the terms containing k-fold integrals describe waves diffracted $k + 1$ times. The composite index (r) of functions $\Pi_{(l)(r)k}$, contained in the expressions under the integral signs gives the past history of the wave described by the corresponding term. For instance, the index pss corresonds to a four-time diffracted wave, which after the first diffraction was propagated as a longitudinal (potential) wave, after the second diffraction — as a transverse (solenoidal) wave, after third diffraction as a transverse wave and is now longitudinal or transverse, depending on the potential expression in which the considered term is contained. Let us note that, according to (2.8), in the multiple integrals taken with respect to the arguments of the functions $L_{(l)s}$ the integration should be carried out from 1 to ∞, whereas with respect to the arguments of the functions $L_{(l)p}$ it should be from γ to ∞, as before. The terms with $k = 0$ represent a special case, since they describe not only the diffracted waves but also those reflected from the strip, and the formation of a shadow as well.

From the form of Expressions (2.3) it can be concluded that two systems of multiply diffracted waves are present. The first system contains waves which were first diffracted at the left edge of the strip (terms in the first braces in (2.3)), and the second system contains those which were first diffracted at the right edge. There exists a simple relation between the two systems, which is easily established by applying theorems of the operational calculus to Formulas (2.3) and (2.1). Namely, let $f(x, y, t, \vartheta)$ be the potential of anyone of the multiply diffracted waves of the first system; the analogous wave of the second system is described by the potential

$$-f(-x, -y, t - 2\vartheta, -\vartheta) \tag{2.10}$$

The above relation does not hold true for waves with $k = 0$

3. In this section let us investigate the once-diffracted and reflected waves. The transforms of potentials in this case are given by Expressions

$$\begin{aligned}\varphi(q, y, p) = \Big[&-\frac{qu_i(p) K_{(1)}(-\vartheta)}{p^2 \sqrt{\gamma^2 p^2 - q^2}\,(\vartheta + q/p)} K_{(1)}\left(-\frac{q}{p}\right) + \\ &+ \operatorname{sgn} y\, \frac{v_i(p) K_{(2)}(-\vartheta)}{p^2(\vartheta + q/p)} K_{(2)}\left(-\frac{q}{p}\right)\Big] \exp\{q - |y| \sqrt{\gamma^2 p^2 - q^2}\} + \\ &+ \Big[\frac{qu_i(p) M_{(1)}(\vartheta)}{p^2 \sqrt{\gamma^2 p^2 - q^2}\,(\vartheta + q/p)} K_{(1)}\left(\frac{q}{p}\right) - \\ &- \operatorname{sgn} y\, \frac{v_i(p) M_{(2)}(\vartheta)}{p^2(\vartheta + q/p)} K_{(2)}\left(\frac{q}{p}\right)\Big] \exp\{-q - |y| \sqrt{\gamma^2 p^2 - q^2}\}\end{aligned} \tag{3.1}$$

$$\begin{aligned}\psi(q, y, p) = \Big[&\operatorname{sgn} y\, \frac{u_i(p) K_{(1)}(-\vartheta)}{p^2(\vartheta + q/p)} K_{(1)}\left(-\frac{q}{p}\right) + \\ &+ \frac{qv_i(p) K_{(2)}(-\vartheta)}{p^2 \sqrt{p^2 - q^2}\,(\vartheta + q/p)} K_{(2)}\left(-\frac{q}{p}\right)\Big] \exp\{q - |y| \sqrt{p^2 - q^2}\} - \\ &- \Big[\operatorname{sgn} y\, \frac{u_i(p) M_{(1)}(\vartheta)}{p^2(\vartheta + q/p)} K_{(1)}\left(\frac{q}{p}\right) + \\ &+ \frac{qv_i(p) M_{(2)}(\vartheta)}{p^2 \sqrt{p^2 - q^2}\,(\vartheta + q/p)} K_{(2)}\left(\frac{q}{p}\right)\Big] \exp\{-q - |y| \sqrt{p^2 - q^2}\}\end{aligned} \tag{3.2}$$

In computing the inversions of (3.2) and (3.1) it is convenient to transfer to a particular case, in which

$$u_i = U_i \delta(t - \vartheta(x + 1) + \delta y), \qquad v_i = V_i \delta(t - \vartheta(x + 1) + \delta y) \tag{3.3}$$

Here one should distinguish the symbol of the delta function from the constant which enters its argument. This inconvenience is caused by the fact that the designations adopted in [1] are used in the above expressions. The constants U_1, V_1 and δ depend on the angle of incidence and the type of wave and are:

$$U_i = -\vartheta, \qquad V_i = \delta = \sqrt{\gamma^2 - \vartheta^2} \tag{3.4}$$

for the case of a longitudinal wave, and

$$U_i = \delta = \sqrt{1 - \vartheta^2}, \qquad V_i = \vartheta \tag{3.5}$$

for a transverse wave. The potentials corresponding to this particular case will be designated by capital letters Φ and Ψ , One can transfer to the general case, in which

$$u_i = U_i f\,(t - \vartheta\,(x + 1) + \delta y), \qquad v_i = V_i f\,(t - \vartheta\,(x + 1) + \delta y) \tag{3.6}$$

by means of Formula

$$\varphi\,(x, y, t) = \int_0^t \Phi\,(x, y, t - \tau)\, f\,(\tau)\, d\tau \tag{3.7}$$

and analogously for Ψ .

Let us start with the computation of $\Psi(x,y,t)$. After the substitution of (3.2) into (2.2) we obtain a linear combination of four inversion integrals. Consider one of them

$$J = \frac{1}{4\pi^2} \int_{-i\infty+c}^{i\infty+c} e^{pt} \int_{-i\infty+c'}^{i\infty+c'} \frac{\exp\{q\,(x + 1) - |\,y\,|\,\sqrt{p^2 - q^2}\}}{p\,(q + \vartheta p)} K_{(1)} \left(-\frac{q}{p}\right) dq\, dp$$

Here we set $c' = 0,\ q = p\sigma.$. Then (3.8)

$$J = \frac{1}{4\pi^2} \int_0^t d\tau \int_{-i\infty+c}^{i\infty+c} e^{p\tau} \int_{l(p)} \frac{\exp\{p\,[\sigma\,(x + 1) - |\,y\,|\,\sqrt{1 - \sigma^2}]\}}{\sigma + \vartheta} K_{(1)}\,(-\sigma)\, d\sigma\, dp$$

where $l(p)$ is the contour in the complex σ-plane, which corresponds to the imaginary axis of the q-plane. Now we set $c = 0$ and decompose the integral with respect to p into a sum of two integrals: from $-i\infty$ to 0 and from 0 to $i\infty$. We obtain

$$J = \frac{1}{4\pi^2} \int_0^t d\tau \left\{ \int_0^{i\infty} dp \int_{l_1} \frac{\exp\{p[\tau + \sigma\,(x + 1) - |\,y\,|\,\sqrt{1 - \sigma^2}]\}}{\sigma + \vartheta} K_{(1)}\,(-\sigma)\, d\sigma + \right.$$

$$\left. + \int_{-i\infty}^{0} dp \int_{l_2} \frac{\exp\{p\,[\tau + \sigma\,(x + 1) - |\,y\,|\,\sqrt{1 - \sigma^2}]\}}{\sigma + \vartheta} K_{(1)}\,(-\sigma)\, d\sigma \right\}$$

where l_1 and l_2 are the contours shown in Fig. 2. The same figure shows the branch cuts going from $-\gamma$ to $-\infty$ and from 1 to ∞ . (Let us remember, that the function $K_{(1)}\,(-\sigma)$ is regular everywhere, except on the segment of the real axis $[-\gamma, -\infty)$). If the order of integration could be interchanged, the

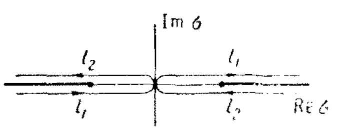

Fig. 2

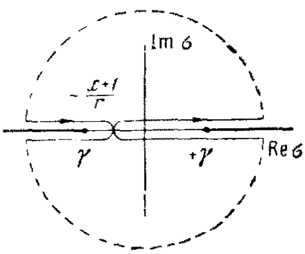

Fig. 3

integral with respect to p would be elementary. To make that possible, the contours l_1 and l_2 should be deformed so that

$$\operatorname{Im}\,[\tau + \sigma(x+1) - |y|\sqrt{1-\sigma^2}]$$

be positive on l_1 and negative on l_2. It turns out that this can be accomplished by shifting the points of intersection of both contours with the real axis to the point $\sigma = -(x+1)/r$, where $r = \sqrt{(x+1)^2 + y^2}$. If $(x+1)/r > \vartheta$, then in deforming the contours one should add the residues at point $\sigma = -\vartheta$, and for $(x+1)/r > \gamma$ also add the integral taken along the edges of the branch cut from $\sigma = -\gamma$ to $\sigma = -(x+1)/r$. Then the integral (3.8) takes the form

$$J = -H(r^{-1}(x+1) - \vartheta)\, H(t - \vartheta(x+1) - |y|\sqrt{1-\theta^2})\, M_{(1)}(\vartheta) +$$

$$+ \frac{1}{\pi} H(r^{-1}(x+1) - \gamma)\, H(r - t)\ \text{v. p.} \int_{\gamma}^{(x+1)/r} H(t - \sigma(x+1) - |y|\sqrt{1-\sigma^2}) \times$$

$$\times \frac{N_{(1)}(\sigma)}{\vartheta - \sigma} d\sigma - \frac{1}{4\pi^2} \int_0^t d\tau \int_{(l_1' - l_2')} \frac{K_{(1)}(-\sigma)\, d\sigma}{(\vartheta + \sigma)\,[\tau + \sigma(x+1) - |y|\sqrt{1-\sigma^2}]} \tag{3.9}$$

Here $H(\tau)$ is the Heaviside's unit step function. We write the symbol of principal value in front of the integral in the second term to take care of the possible case $\vartheta > \gamma$.

Moreover, as in [1],

$$N_{(l)}(\sigma) = L_{(l)}(\sigma)\, K_{(l)}(-\sigma)$$

Let us now consider the last term. The contour $(l_1' - l_2')$ is shown in Fig.3. We close it in the upper and lower half-planes by semicircles of infinitely large radii, as shown schematically in the figure, and thus reduce the integral with respect to σ to a sum of residues at the poles.

$$\sigma = \sigma_{1,2} = r^{-2}\,[-\tau(x+1) \pm i\,|y|\sqrt{\tau^2 - r^2}]$$

bearing in mind that these poles fall inside the integration contours only for $\tau > r$.

Thus, the last term (3.9) becomes

$$-\frac{1}{2\pi} \int_r^t \left\{ \frac{K_{(1)}(-\sigma_1)\sqrt{1-\sigma_1^2}}{\vartheta + \sigma_1} + \frac{K_{(1)}(-\sigma_2)\sqrt{1-\sigma_2^2}}{\vartheta + \sigma_2} \right\} \frac{d\tau}{\sqrt{\tau^2 - r^2}}$$

Since the terms in braces are complex conjugate, the expression under the integral sign is real, and the integral (3.8) is reduced to quadratures. However, for obtaining the asymptotic expansions in the neighborhood of the wave fronts, it is convenient to transfer the last expression again to a certain contour integral in the σ-plane. This is accomplished by a simple change of variables. Finally, we obtain

$$J = -H(r^{-1}(x+1) - \vartheta)\, H(t - \vartheta(x+1) - |y|\sqrt{1-\vartheta^2})\, M_{(1)}(\vartheta) +$$

$$+ \frac{1}{\pi} H\left(\frac{x+1}{r} - \gamma\right) H(r - t)\ \text{v.p.} \int_{\gamma}^{(x+1)/r} H(t - \sigma(x+1) -$$

$$- |y|\sqrt{1-\sigma^2})\, \frac{N_{(1)}(\sigma)}{\vartheta - \sigma} d\sigma - \frac{1}{2\pi i} \int_{L_s} \frac{K_{(1)}(\sigma)\, d\sigma}{\vartheta - \sigma} H(t - r) \tag{3.10}$$

Here L_s is the contour passing from point

$$\sigma = r^{-2}\,[t(x+1) - i\,|y|\sqrt{t^2 - r^2}]$$

to point $\sigma = r^{-2}\,[t(x+1) + i\,|y|\sqrt{t^2 - r^2}]$ and intersecting the real axis at $\sigma = r^{-1}(x+1)$.

The remaining three integrals which appear in the expression for Ψ are analogous to (3.8). Thus

$$\Psi(x, y, t) = \operatorname{Re} \frac{\operatorname{sgn} y U_i \sqrt{\gamma^2 - \vartheta^2} - V_i \vartheta}{\vartheta^2 + \sqrt{1 - \vartheta^2}\sqrt{\gamma^2 - \vartheta^2}} H(\cos\alpha^- - \vartheta) H(t -$$

$$- \vartheta\xi^- - |y| \sqrt{1 - \vartheta^2}) H(\cos\alpha^+ + \vartheta) + \tag{3.11}$$

$$+ \frac{1}{2\pi i} H(t - r^-) \int_{L_s^-} \frac{\operatorname{sgn} y\, U_i K_{(1)}(-\vartheta) K_{(1)}(\sigma)\sqrt{1-\sigma^2} - V_i K_{(2)}(-\vartheta) K_{(2)}(\sigma)\sigma}{(\vartheta - \sigma)\sqrt{1-\sigma^2}} d\sigma +$$

$$+ \frac{1}{2\pi i} H(t^+ - r^+) \int_{L_s^+} \frac{\operatorname{sgn} y U_i M_{(1)}(\vartheta) K_{(1)}(\sigma)\sqrt{1-\sigma^2} + V_i M_{(2)}(\vartheta) K_{(2)}(\sigma)\sigma}{(\vartheta + \sigma)\sqrt{1-\sigma^2}} d\sigma +$$

$$+ \frac{1}{\pi} H(r^- - t) H(\cos\alpha^- - \gamma) H(\cos\alpha^+ + \gamma) \text{ v. p.} \int_\gamma^{\cos\alpha^-} H(t - \sigma\xi^- -$$

$$- |y|\sqrt{1-\sigma^2}) \frac{\operatorname{sgn} y\, U_i K_{(1)}(-\vartheta) N_{(1)}(\sigma)\sqrt{1-\sigma^2} - V_i K_{(2)}(-\vartheta) N_{(2)}(\sigma)\sigma}{(\vartheta - \sigma)\sqrt{1-\sigma^2}} d\sigma +$$

$$+ \frac{1}{\pi} H(r^+ - t^+) H(\cos\alpha^- + \gamma) H(\cos\alpha^+ - \gamma) \int_\gamma^{\cos\alpha^+} H(t^+ - \sigma\xi^+ -$$

$$- |y|\sqrt{1-\sigma^2}) \frac{\operatorname{sgn} y\, U_i M_{(1)}(\vartheta) N_{(1)}(\sigma)\sqrt{1-\sigma^2} + V_i M_{(2)}(\vartheta) N_{(2)}(\sigma)\sigma}{(\vartheta + \sigma)\sqrt{1-\sigma^2}} d\sigma$$

Analogously, for $\Phi(x,y\ t)$ we obtain Expression

$$\Phi(x, y, t) = \operatorname{Re} \frac{U_i\vartheta + \operatorname{sgn} y V_i \sqrt{1-\vartheta^2}}{\vartheta^2 + \sqrt{1-\vartheta^2}\sqrt{\gamma^2 - \vartheta^2}} H(\gamma\cos\alpha^- - \vartheta) H(\gamma\cos\alpha^+ +$$

$$+ \vartheta) H(t - \vartheta\xi^- - |y|\sqrt{\gamma^2 - \vartheta^2}) + \tag{3.12}$$

$$+ \frac{1}{2\pi i} H(t - \gamma r^-) \int_{L_p^-} \frac{U_i K_{(1)}(-\vartheta) K_{(1)}(\sigma)\sigma + \operatorname{sgn} y V_i K_{(2)}(-\vartheta) K_{(2)}(\sigma)\sqrt{\gamma^2-\sigma^2}}{(\vartheta - \sigma)\sqrt{\gamma^2 - \sigma^2}} d\sigma +$$

$$+ \frac{1}{2\pi i} H(t^+ - \gamma r^+) \int_{L_p^+} \frac{U_i M_{(1)}(\vartheta) K_{(1)}(\sigma)\sigma - \operatorname{sgn} y V_i M_{(2)}(\vartheta) K_{(2)}(\sigma)\sqrt{\gamma^2-\sigma^2}}{(\vartheta + \sigma)\sqrt{\gamma^2 - \sigma^2}} d\sigma$$

The following designations have been introduced in Formulas (3.11) and (3.12):

$$t^+ = t - 2\vartheta, \qquad \xi^- = r^-\cos\alpha^- = x + 1, \qquad r^- = \sqrt{(\xi^-)^2 + y^2}$$

$$\xi^+ = r^+\cos\alpha^+ = 1 - x, \quad r^+ = \sqrt{(\xi^+)^2 + y^2}$$

the contour L_s^- passes from point $\sigma = (r^-)^{-2}(t\xi^- - i|y|\sqrt{t^2 - \gamma^2(r^-)^2})$ to point $\sigma = (r^-)^{-2}(t\xi^- + i|y|\sqrt{t^2 - \gamma^2(r^-)^2})$, intersecting the real axis at $\sigma = \gamma\cos\alpha^-$, the contour L_s^- has been defined above, and the contours L_p^+ and L_s^+ are determined in the same way as L_p^- and L_s^-, but in terms of quantities designated by the sign $^+$.

Now it is easy to establish the physical meaning of each term in Expressions (3.11) and (3.12). The first term describe the reflected longitudinal and transverse waves for $y > 0$, and for $y < 0$ they cancel the incident wave, thus securing the formation of the geometric shadow. The next two terms in the expressions of each potential describe cylindrical waves diffracted at the left and right edges of the strip. We will symbolically designate those waves by tp^-, tp^+, ts^- and ts^+. (The symbol p indicates longitudinal (potential) waves, symbol s – transverse (solenoidal) waves). The last two terms in (3.11) represent the potentials of the head waves. Those will be designated by is^{-*} and is^{+*}. Fig.4 shows the location of fronts of all those waves for the time interval $2\vartheta < t < 2\gamma$.

4. Expressions (3.11) and (3.12) completely describe the diffraction for $0 < t < 2\gamma$, i.e. up to the instant at which the wave front ip^- reaches the right edge of the strip. At that instant the waves ip^-p^+, ip^-s^+, ip^-s^{+*} appear. In the designations of multiply diffracted waves the sequence of indices is determined by the wave's history: the index i designates the incident wave, indices p and s with the $^-$ or $^+$ sign designate diffraction at the left or right edge of the strip, respectively; the asterisk at the index s designates the head wave. Clearly, the index with an asterisk can appear in the last place only, whereas the indices with $^-$ or $^+$ signs alternate. Hence, it is sufficient to show which of the two indices with $^-$ or $^+$ signs follows the incident wave index, i.e. instead of $ip^-p^+s^-$ or $ip^+p^-s^+$ we will write ip^-ps or ip^+ps. In the general case we will write $i(r^-)s$, $i(r^+)s$ or $i(r^-)p$. Formula (2.10) of Section 2 gives a simple relation between the potentials of waves $i(r^-)p$, $i(r^-)s$ and $i(r^+)p$, $i(r^+)s$. We should note, however, that the substituting ϑ for $-\vartheta$ one should write $M_{(l)}(\vartheta)$ instead of $K_{(l)}(-\vartheta)$, and vice versa. Therefore, we will confine ourselves to the computation of wave potentials for $i(r^-)p$, $i(r^-)s$ and $i(r^-)s^*$. From (2.1), (2.3) and (2.9) we obtain the expression for the transform of the wave potential of $i(r^-)p$:

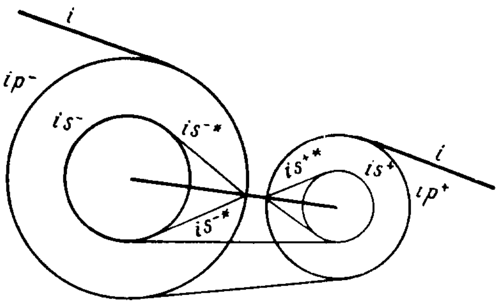

Fig. 4

$$\varphi = \Big[\frac{qu_i(p)}{p^2\sqrt{\gamma^2p^2-q^2}}K_{(1)}\Big((-1)^{k+1}\frac{q}{p}\Big)\exp[(-1)^k q]\quad \text{v. p.}\int\limits_{\Delta_{(r)}}\times$$

$$\times\,\Pi_{(1)(r)k}(\zeta)\frac{d\Omega_\zeta}{(\zeta_1-\vartheta)\,[\zeta_k-(-1)^k q/p]}-\operatorname{sgn} y\,\frac{v_i(p)}{p^2}K_{(2)}\Big((-1)^{k+1}\frac{q}{p}\Big)\times$$

$$\times\exp[(-1)^k q]\quad \text{v. p.}\int\limits_{\Delta_{(r)}}\Pi_{(2)(r)k}(\zeta)\frac{d\Omega_\zeta}{(\zeta_1-\vartheta)\,[\zeta_k-(-1)^k q/p]}\Big]\times$$

$$\times\exp\{-|y|\sqrt{\gamma^2p^2-q^2}\} \tag{4.1}$$

where $k+1$ is the multiplicity of diffraction, $\Delta_{(r)}$ is the domain of integration which is determined as follows: if in the composite index (r) the j-th place is occupied by the index p, the integration will respect to ζ_j is carried out from γ to ∞, otherwise it is from 1 to ∞.

It is easy to verify that in the process of inversion it is permissible to perform integration under the multiple integral sign without restrictions. Moreover, the resulting integrals are again of (3.8) type, and thus we obtain (again for the case in which the incident wave has the form (3.3))

$$\Phi(x,y,t)=\Big[(-1)^k K_{(1)}(-\vartheta)\quad \text{v. p.}\int\limits_{\Delta_{(r)p}}\frac{\Pi_{(1)(r)k}{}^*(\zeta)}{\zeta_1-\vartheta}\int\limits_{L_{kp}}\frac{K_{(1)}(\sigma)\,\sigma\,d\sigma}{(\zeta_k+\sigma)\sqrt{\gamma^2-\sigma^2}}\,d\Omega_\zeta U_i\,-$$

$$-\operatorname{sgn} yK_{(2)}(-\vartheta)\quad \text{v. p.}\int\limits_{\Delta_{(r)p}}\frac{\Pi_{(2)(r)k}{}^*(\zeta)}{\zeta_1-\vartheta}\int\limits_{L_{kp}}\frac{K_{(2)}(\sigma)\,d\sigma}{\zeta_k+\sigma}\,d\Omega_\zeta V_i\Big]\frac{1}{2\pi i} \tag{4.2}$$

where $\Delta_{(r)p}$ is the domain of integration determined by conditions

$$\Delta_{(r)p}\subset\Delta_{(r)},\qquad t_k\equiv t-2\sum_{j=1}^{k}\zeta_j>\gamma r_k\qquad (r_k=\sqrt{\xi_k{}^2+y^2},\ \xi_k=1+(-1)^k x)$$

The contour L_{kp} in the σ-plane is defined in terms of t_k, ξ_k, r_k in the same way as the contour L_p^- in (3.12) was defined in terms of t, ξ^- and r^-, and also

$$\Pi^*_{(l)(r)k}(\zeta) = \Pi_{(l)(r)k}(\zeta)\exp\left\{2p\sum_{j=1}^{k}\zeta_j\right\} \tag{4.3}$$

i.e. it is obtained from $\Pi_{(l)(r)k}$ by discarding the exponential factor. The condition $\Delta_{(r)p} \subset \Delta_{(r)}$ can be written down in the form

$$\zeta_j \geqslant \zeta_{j0}\begin{cases}\zeta_{j0} = \gamma, & \text{if } p \text{ appears in } j\text{th place in } (r)\\ \zeta_{j0} = 1, & \text{if } s \text{ appears in } j\text{th place in } (r)\end{cases}$$

Suppose the index p is contained in (r) m times, and the index s $(k-m)$ times. Then, as we readily find, the wave $t(r^-)p$ appears when $t = 2m\gamma + 2(k-m)$, as should be expected from kinematic considerations. For the potential of the wave $t(r^-)s$ we obtain

$$\Psi(x, y, t) = \Big[\operatorname{sgn} y U_i K_{(1)}(-\vartheta)\ \text{v. p.}\int\limits_{\Delta_{(r)s}}\frac{\Pi^*_{(1)(r)k}(\zeta)}{\zeta_1-\vartheta}\int\limits_{L_{ks}}\frac{K_{(1)}(\sigma)\,d\sigma}{\zeta_k+\sigma}\,d\Omega_\zeta -$$

$$-(-1)^k V_i K_{(2)}(-\vartheta)\ \text{v. p.}\int\limits_{\Delta_{(r)s}}\frac{\Pi^*_{(2)(r)k}(\zeta)}{\zeta_1-\vartheta}\int\limits_{L_{ks}}\frac{K_{(2)}(\sigma)\,\sigma d\sigma}{\sqrt{1-\sigma^2}\,(\zeta_k+\sigma)}\,d\Omega_\zeta\Big]\frac{i}{2\pi} \tag{4.4}$$

and the expression for the potential of the wave $t(r^-)s^*$ is

$$\Psi(x, y, t) = -\frac{1}{\pi}H\Big(r_k - t + 2\sum_{j=1}^{k}\zeta_{j0}\Big)H(\cos\alpha_k-\gamma)\,H(\cos\alpha_{k+1}+\gamma)\times$$

$$\times\Big[\operatorname{sgn} y K_{(1)}(-\vartheta)\,U_i\ \text{v. p.}\int\limits_{\Delta_{(r)p}}\frac{\Pi^*_{(1)(r)k}(\zeta)}{\zeta_1-\vartheta}\int\limits_{\gamma}^{\cos\alpha_k}H(\lambda_k)\frac{N_{(1)}(\sigma)\,d\sigma}{(\zeta_k+\sigma)}\,d\Omega_\zeta - \tag{4.5}$$

$$-(-1)^k K_{(2)}(-\vartheta)\,V_i\ \text{v. p.}\int\limits_{\Delta_{(r)p}}\frac{\Pi^*_{(2)(r)k}(\zeta)}{\zeta_1-\vartheta}\int\limits_{\gamma}^{\cos\alpha_k}H(\lambda_k)\frac{N_{(2)}(\sigma)\,\sigma d\sigma}{\sqrt{1-\sigma^2}\,(\zeta_k+\sigma)}\,d\Omega_\zeta\Big]$$

where

$$\cos\alpha_k = \frac{\xi_k}{r_k}, \qquad \lambda_k = t_k - \sigma\xi_k - |y|\sqrt{1-\sigma^2} \tag{4.6}$$

the contour L_{ks} is defined in terms of t_k, ξ_k, r_k analogously to the contour L_s^-, and the domain of integration L_s^-, is determined by conditions

$$\Delta_{(r)s} \subset \Delta_{(r)}, \qquad t_k > r_k \tag{4.7}$$

The reasons for the appearance of the factor $H(\cos\alpha_{k+1}+\gamma)$ in Expression (4.5) are not obvious. Indeed, in the inversion of the kth term in Formula (2.3) there appears a term which is due to the residue at point $\sigma = \zeta_k$ and which is different from zero only for $\cos\alpha_k < -\zeta_k$

After some transformations it becomes clear that this term cancels the head wave which has appeared after the diffraction of the opposite edge of the strip, in the region shown hatched in Fig.5, and should be referred to $(k-1)$th term, while on analogous item from the $(k+1)$th term should be added to the kth potential. The appearance of analogous factors in the potentials of the head waves in Formula (3.11) is due to trasfer of analogous factors from the terms with $k = 1$

5. To describe the disturbances arising as a result of motion of the strip one could proceed from formulas for transforms (analogous to Formula (2.3)) from the paper [1], but there is a shorter way available. Indeed, from the form of the boundary conditions (1.2) we can conclude that the disturbances generated by the translation of the strip must be the same as in the case of diffraction of a longitudinal incident wave with the displacement $v_i = -v_0(t - \gamma y)$ and a transverse incident wave with the displacement $u_i = -u_0(t - y)$ both falling normally upon the fixed strip. It is clear that the number of individual waves generated in translation of the strip is identical to that arising in diffraction at the fixed strip. It is easy to transfer from formulas describing potentials of waves arising in the diffraction of a wave of form (3.3) to the potentials of corresponding waves generated by the strip's translation

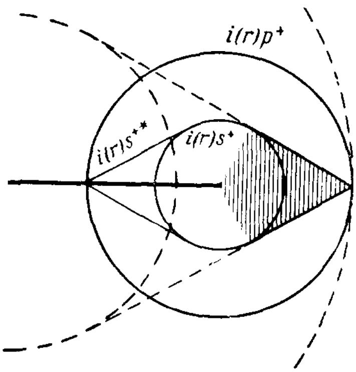

Fig. 5

$$\varphi_0(x, y, t) = -\int_0^t [u_0(t-\tau)\Phi_1 + v_0(t-\tau)\Phi_2]d\tau$$

$$\Phi = \Phi_1(x, y, \tau;\ U_i = 1,\ V_i = 0,\ \vartheta = 0)$$

$$\Phi = \Phi_2(x, y, \tau;\ U_i = 0,\ V_i = 1,\ \vartheta = 0)] \qquad (5.1)$$

and analogously for potentials of transverse waves.

In order to transfer from formulas describing the diffraction of waves (3.3) of the fixed strip to those describing the disturbance generated by the strip's rotation, it is sufficient to introduce the derivative with respect to the parameter ϑ. Then, it is easy to verify that the corresponding expressions for the potentials must be of the form

$$\varphi_\alpha = \int_0^t \left\{\left[\frac{\partial}{\partial\vartheta}\Phi(x, y, \tau;\ U_i = 0,\ V_i = 1, \vartheta)\right]_{\vartheta=0} \int_0^{t-\tau} \alpha(\tau_1)\, d\tau_1\right\} d\tau +$$

$$+ (1 - x_0)\int_0^t \alpha(t-\tau)\,\Phi(x, y, \tau;\ U_i = 0,\ V_i = 1,\ \vartheta = 0)\, d\tau$$

We will not dwell on the question of these waves in more detail, since near the wave fronts they are asymptotically small as compared to the diffracted waves (namely, as is readily seen from formulas for the strip's displacement and rotation in [1], the discontinuities of these waves at the wave fronts, as compared to the discontinuities of the diffracted waves at the wave fronts are lower by one order of magnitude for the case of the right angle of incidence ($\vartheta = 0$), and by two orders of magnitude in other cases (for $\vartheta \neq 0$)).

The formulas obtained in Section 3 to 5 are cumbersome and inconvenient for practical applications. Therefore, it would be expedient to obtain asymptotic expressions describing the diffracted waves near their fronts and at large distances away from the strip. However, for that it would be necessary to investigate a great number of particular cases which would be beyond the scope of the present paper.

We would like to point out that a closely related problem of the formation of a crack having the shape of a strip, in a prestressed elastic medium has been considered by Flitman [2].

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